Tower of PISA

Standards Addressed

1. The Standards for Mathematical Practice, especially: **1. Make sense of problems and persevere in solving them** and 2. Reason abstractly and quantitatively.
2. 8.G.B.5: Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.
3. G.SRT.B.5 & M2G.SRT.B.5: Use congruence and similarity criteria for triangles to solve problems and to justify relationships in geometric figures. (Also B.G.SRT.)

Before conducting the activity

1. Reproduce the two work sheets ("Seeing the Tower" and "How Far?").
2. Think through the solution you would like your students to discover to the second work sheet. In pre-algebra classes, you might expect students to draw the picture and estimate the distances with the ruler (which is scaled to match the tower's inscribed radius). In Algebra I, using the Pythagorean Theorem and similar triangles, students can calculate the distance both exactly (symbolically) and then approximately numerically. Late in in Algebra II (or in precalculus), you students can use trigonometry.
3. Review the necessary material with your students (depends on your chosen approach: similar triangles, Pythagorean Theorem, trigonometry definitions?)

Conducting the Activity

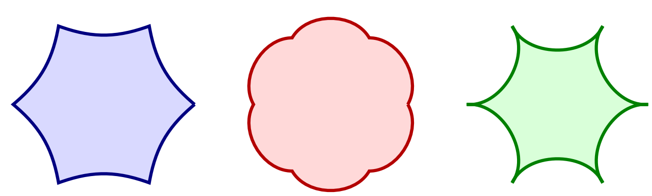
1. Give each student a copy of the page titles "Seeing the Tower" and have them complete the activity.
2. Have the students compare answers in groups and then ask the groups to explain how to solve this problem. (Their answer should include extending the sides into lines.)
3. Have the students work in groups on the activity title "How Far?" Before they start, make sure they understand what is being asked. I would limit my explanation and examples to points outside of the octagonal tower—let them think of being inside the tower on their own. (More detailed suggestions are in the Teacher's Guide.)
4. As a class, compare both the groups answers and their methods of solution.

Variations

You could alter the number of sides of the regular n-gon tower. Both $n = 3$ and $n = 4$ are very easy and require very little math. The case $n = 6$ is a little easier than the case $n = 8$ presented here. However, $n = 5$, $n = 7$ and $n = 9$ are quite difficult and (essential) require trigonometry or just using a ruler to estimate distances.

Extension

Use circular arcs to define your tower.



Seeing the Tower* (PISA Exam 2012)

In Figures 1 and 2 below, you see two drawings of the same tower. In Figure 1 you see three faces of the roof of the tower. In Figure 2 you see four faces. In the following diagram, the view of the roof of the tower, from above, is shown. Five positions are shown on the diagram. Each is marked with a cross (×) and they are labeled P1–P5.

Figure 1

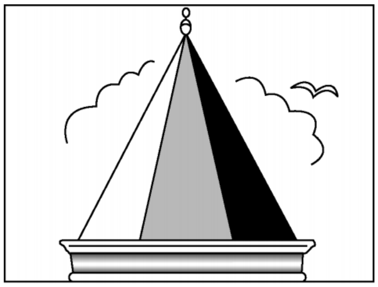
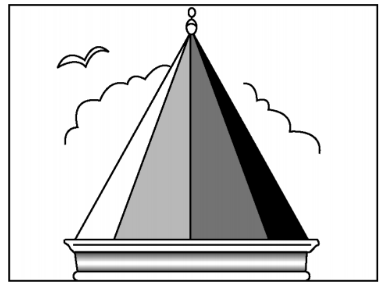
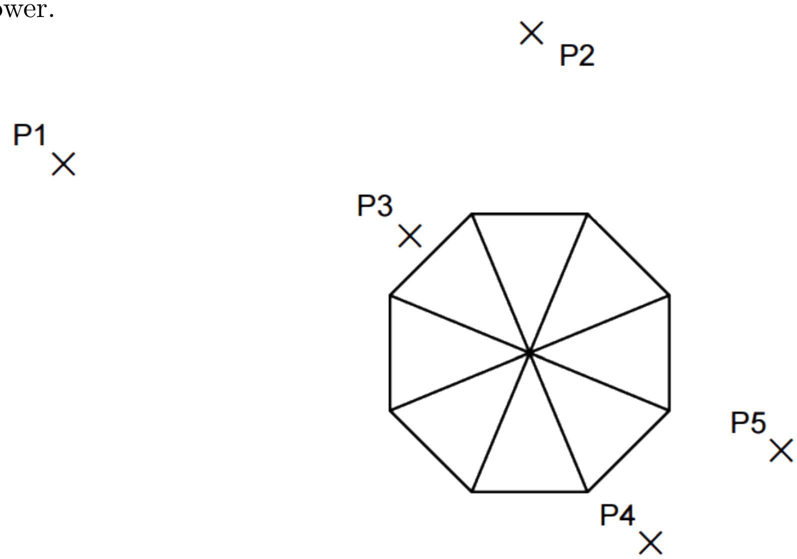


Figure 2



From each of these positions, a person viewing the tower would be able to see a number of faces of the roof of the tower.



In the table below, circle the number of faces that could be seen from each of these positions.

Position	Number of faces that could be seen (circle the correct number)				
P1	1	2	3	4	more than 4
P2	1	2	3	4	more than 4
P3	1	2	3	4	more than 4
P4	1	2	3	4	more than 4
P5	1	2	3	4	more than 4

*Source `https://nces.ed.gov/surveys/pisa/pdf/items2_math2012.pdf` June 2016.

How Far?

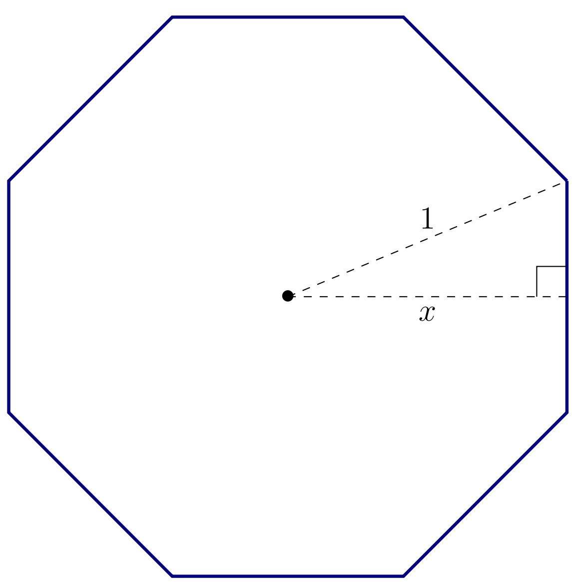
What is the closest that you can be to the center of the tower and see exactly 1 side? exactly 2 sides? exactly 3 sides? ...

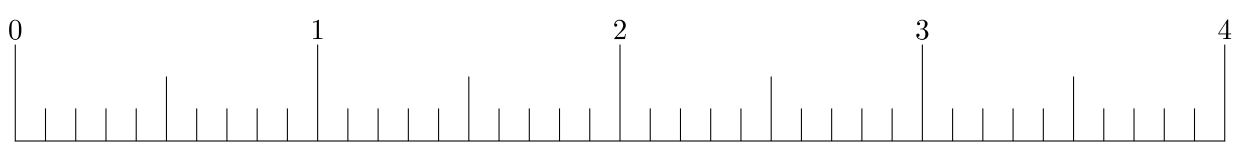
What are the furthest that you can be from the center and see exactly 1 side? 2 sides? 3 sides?...

Fill our the chart on the right with your answers and be ready to explain your group's solutions to the class.

Look at the scale in the drawing (especially where 1 unit is) and write the answers in term of the distance x units.

sides	closest	furthest
1		
2		
3		
4		
5		
6		
7	not possible	not possible
8		

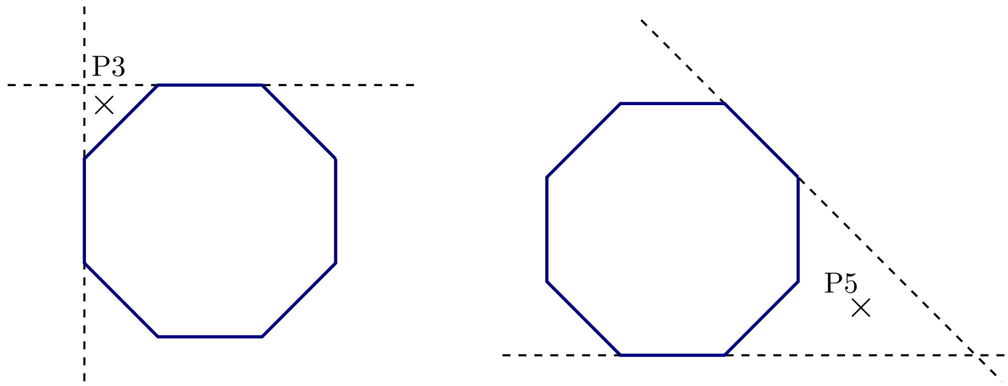




Teacher's Guide

Seeing the Tower

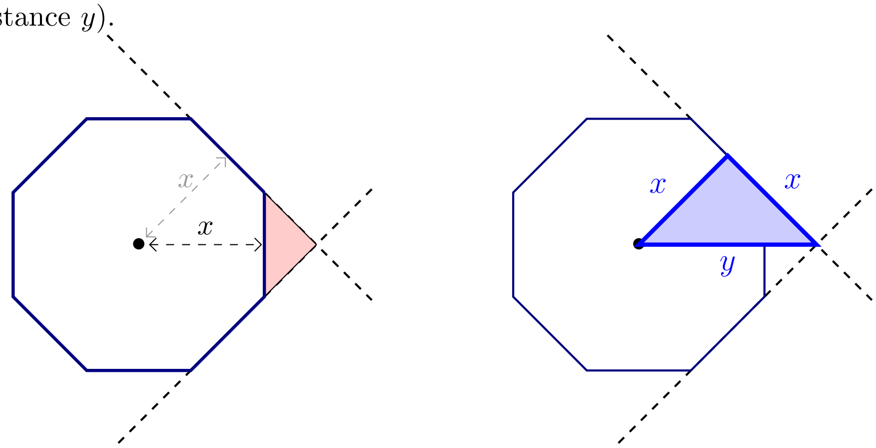
The correct answers are 4, 3, 1, 2, 2 respectively. It is unlikely a student will get them all correct without understanding the method. It is also unlikely that they will all see the method on their own. Let the students persevere for awhile, then ask them "pick a side of the tower, where must you be standing to see that side?" (Answer: if you extend that side into a line, you must be standing on the side of the line opposite of the tower.) If this is not a suffice hint, then discuss person P3 or P5, ask what sides of the tower they can see and why..



How Far?

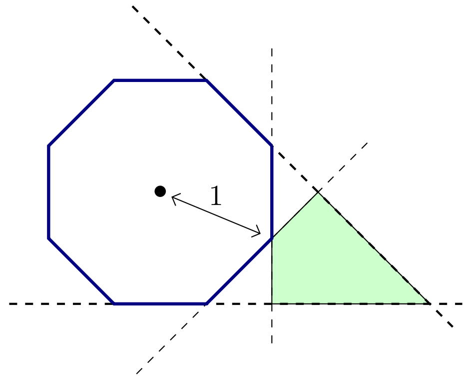
Once the student groups have worked on this problem for awhile (recall that making sense of problems and persevering in solving them is the first, and perhaps most important, of the standards for mathematical practice–so always model and encourage persistence). Finding x exactly is usually done with the half angle formula for cosine ($x = \cos 22.5°$ which is exactly $(\sqrt{2+\sqrt{2}})/2$ and approximately 0.9239). However it is both sufficient and pedagogically advantageous to just call it x for now. The heart of algebra is the abstraction involved in manipulating x.

Once they have struggled enough, pause their work and work through the case of one side with them. **The closet we can be and see exactly one side is clearly x units** (because any closer and we are in the tower). To find the furthest we can be away, extend two sides as follows (and call the longest distance y).

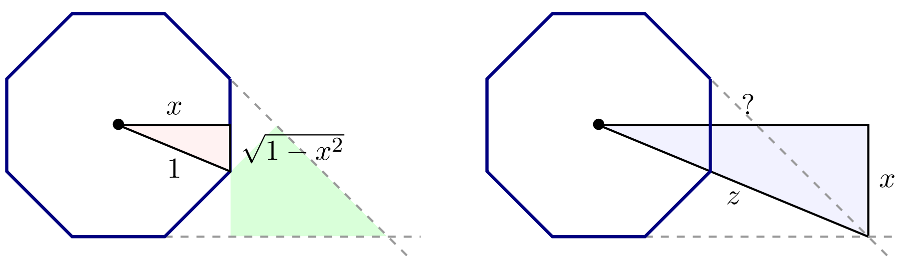


The class should easily see **the furthest away one can be and see exactly one side is** $y = \sqrt{2}x$ **units.** We have completed the first row (and about 40% of the work).

Again let the class stew on seeing exactly two sides. They should be able to draw an appropriate region by extending two sides as shaded below. Now the closest you can be and see exactly two sides can be found with very little thought (it is the radius o the circumscribed circle). (**The closest you could see exactly two sides is 1 unit!**) so don't give that information to them directly (stick with hints even when giving the answer is so much easier).



Call the furthest distance for seeing two sides z. Notice we have two similar right triangles, the one inside the octagon with hypotenuse 1, side x and therefore side $\sqrt{1-x^2}$; and the one with hypotenuse z and vertical side x (the other side labelled '?' is actually $x + y$).



This give the equal ratios

$$\frac{z}{x} = \frac{1}{\sqrt{1-x^2}},$$

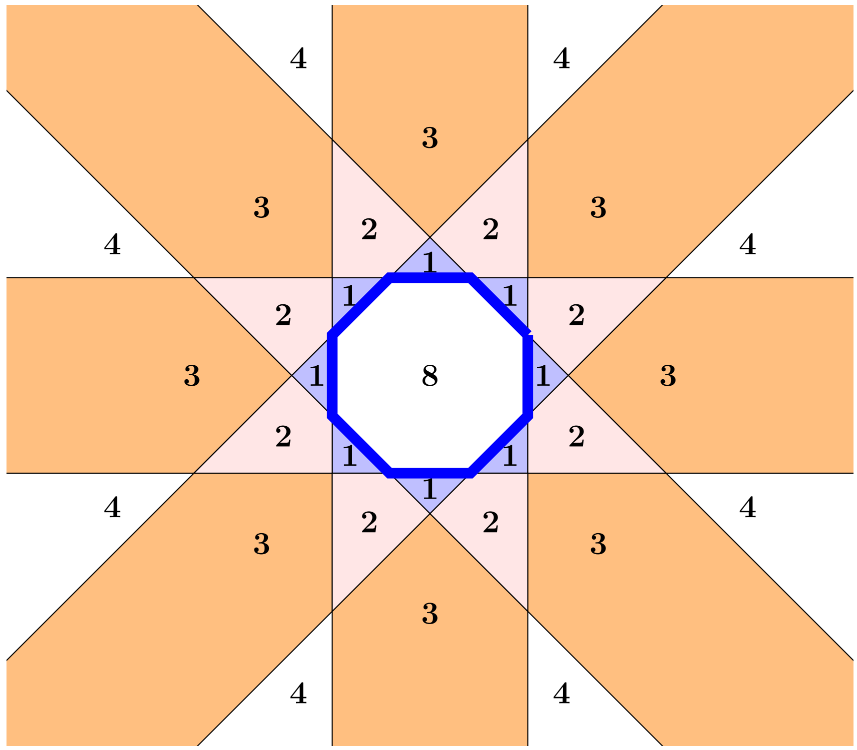
so **the furthest away that you could see exactly two sides is** $z = \frac{x}{\sqrt{1-x^2}} \approx 3.414$**.**

We have now completed just the first two rows of the table (and over 90% of the work), but the rest are so easy that no more calculations (and letters) are needed. Here is what the complete table will look like.

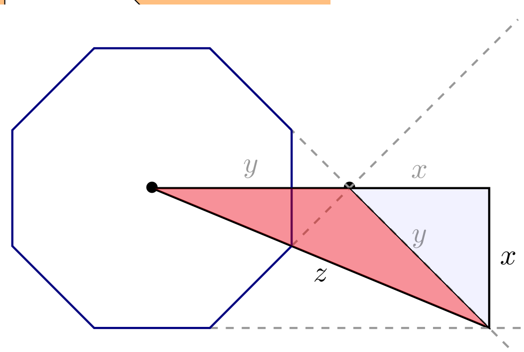
sides	closest	furthest
1	x	y
2	1	z
3	y	infinity
4	z	infinity
5,6,7	not possible	not possible
8	0	1

In this table $x = \cos 22.5° \approx 0.9239$, $y = \sqrt{2}\,x \approx 1.3066$, and $z = \frac{x}{\sqrt{1-x^2}} \approx 2.4142$. (You might guess from the approximation that $z = 1 + \sqrt{2}$ and you would be correct!)

Now just look at the complete picture and you can see the rest of the entries in the table came from.



Bonus: A way to find x



I would skip this part in almost all classes, but wanted to point out the figure with the triangle that we used to find z (slightly altered) also allows us to solve for x. First notice the horizontal side of the triangle labelled '?' actually has length $x + y = (1+\sqrt{2})x$.

By the Pythagorean Theorem, $z^2 = x^2 + (1+\sqrt{2})^2x^2 = (4+2\sqrt{2})x^2$. Earlier we saw $z = \frac{x}{\sqrt{1-x^2}}$, so we can equate two formulas for z^2.

$$z^2 = \frac{x^2}{1-x^2} = (4+2\sqrt{2})x^2.$$

Divide the right equality by x^2 and invert to get

$$1 - x^2 = \frac{1}{4+2\sqrt{2}},$$

so

$$x^2 = 1 - \frac{1}{4+2\sqrt{2}} = \frac{3+2\sqrt{2}}{4+2\sqrt{2}}.$$

The denominator of this last fraction is $2(2+\sqrt{2})$ and the numerator is $\frac{1}{2}(2+\sqrt{2})^2$ (just multiply it out an see!), so we get

$$x^2 = \frac{\frac{1}{2}(2+\sqrt{2})^2}{2(2+\sqrt{2})} = \frac{2+\sqrt{2}}{4}$$

and finally $x = \dfrac{\sqrt{2+\sqrt{2}}}{2}$ as advertised earlier (but without trigonometry and a half-angle formula).