IS $\pi(6521) = 6! + 5! + 2! + 1!$ UNIQUE?

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The prime counting function, $\pi(x)$, counts exactly how many primes there are less than or equal to x. The second author discovered the following "curio" (see [1]):

$$\pi(6521) = 6! + 5! + 2! + 1!.$$

If we write the positive integer x in base 10:

$$x = a_k \dots a_2 a_1 a_0 \qquad \text{(with } a_k \geq 0\text{)}$$

are there any other prime solutions to

$$f(x) := \sum_{i=0}^{k} a_i! = \pi(x)\ ? \tag{1}$$

How many solutions could be generated if we allow x to be composite? Is there an upper bound on how far we would need to look? What if we work in a base other than 10 or use other functions? Below we provide answers to these questions, and then pose new areas for further investigation.

Searching for another

By the prime number theorem [2, pp. 225-227], the prime counting function $\pi(x)$ is asymptotic to $x/\ln x$. In fact, Dusart [3] has shown that, when $x \geq 599$,

$$\frac{x}{\ln x}\left(1 + \frac{0.992}{\ln x}\right) < \pi(x) < \frac{x}{\ln x}\left(1 + \frac{1.2762}{\ln x}\right). \tag{2}$$

The factorial $a_i!$ is at most 9! for each of the [1+log x] digits of x, so any solution x to (1) must satisfy

$$\frac{x}{\ln x}\left(1+\frac{0.992}{\ln x}\right) < \pi(x) = \mathrm{f}(x) \le 9!\left[1+\frac{\ln x}{\ln 10}\right]. \tag{3}$$

This statement is false for $x > 48{,}657{,}759$, so this is an upper bound for solutions. If x is an eight-digit solution beginning with 4, then the second digit is at most 8 and we can use the tighter bound

$$\mathrm{f}(x) \le 4! + 8! + 9!\,6 < \pi(40{,}000{,}000) = 2{,}433{,}654$$

to see that there are no such solutions. Now we know $x < 40{,}000{,}000$. After checking to see that 39,999,999 does not work, we note that for $N_1 = (3.8)10^7 \le x < 39{,}999{,}999$ we have

$$\mathrm{f}(x) \le 3! + 8! + 9!\,6 < \pi(N_1) = 2{,}318{,}966.$$

Similarly for $N_2 = (3.6)10^7 \le x < N_1$ we have

$$\mathrm{f}(x) \le 3! + 7! + 9!\,6 < \pi(N_2) = 2{,}204{,}262.$$

Therefore there are no solutions with $x \ge N_2$.

For $N_3 = (3.0)10^7 \le x < N_2$, first we check the cases where x ends in six '9's individually; then for the remaining integers x we have

$$\mathrm{f}(x) \le 3! + 5! + 8! + 9!\,5 < \pi(N_3) = 1{,}857{,}859.$$

A check of the integers $x \le N_3$ using the public domain program UBASIC [4] shows the following 23 solutions:

6500, 6501, 6510, 6511, **6521**, 12066, 50372, 175677, 553783, **5224903**, 5224923, 5246963, 5302479, 5854093, 5854409, 5854419, 5854429, 5854493, 5855904, 5864049, 5865393, 10990544, 11071599 [5, seq. A049529].

Of these, only 6,521 and 5,224,903 are prime [6, p. 11].

Bases other than 10

We can write x in a base B other than 10

$$x = b_k \dots b_2 b_1 b_0 \qquad \text{(with } b_k > 0\text{)}$$

and ask whether the equation

$$\mathrm{g}(x) \;:=\; \sum_{i=0}^{k} b_i! \;=\; \pi(x) \tag{4}$$

has any solutions. Now $b_i! \le (B-1)!$ so we can replace the inequality (3) with

$$\frac{x}{\ln x} \;<\; \pi(x) \;=\; \mathrm{g}(x) \;\le\; (B-1)!\left[1+\frac{\ln x}{\ln B}\right]. \tag{5}$$

Omitting the factor $1+0.992/\ln x$ from (3) ensures that the leftmost inequality holds for $x \ge 11$ rather than $x \ge 599$.

For each value of B the right side of (5) grows like a multiple of $\ln x$, whereas the left-hand side grows like $x/\ln x$, therefore the inequality is false for all large x. So there is a value $x_o(B)$ such that any solution satisfies $x \le x_o(B)$. We will show that we can take $x_o(B) = 2\,B\,B!\,\ln B$ for all bases $B > 2$. Since (5) is already false at $x = 13$ for $B = 2$, we may take $x_o(2) = 13$.

First note for any solution x we have $x \ge B$ (otherwise $x! = \pi(x)$), so (5) yields

$$\frac{x}{\ln x} \;<\; (B-1)!\left(1+\frac{\ln x}{\ln B}\right) \le \frac{2\,(B-1)!\,\ln x}{\ln B}\,. \tag{6}$$

We next show that $x < B^B$ (for $B \ge 3$). Otherwise, since $x/(\ln x)^2$ is an increasing function for $x > e^2$, the inequality above divided by $\ln x$ gives:

$$\frac{B^{\boldsymbol{B}}}{B^2(\ln B)^2} \le \frac{x}{(\ln x)^2} < \frac{2\,(B-1)!}{\ln B} < \frac{2B}{\ln B}\left(\frac{B}{\mathrm{e}}\right)^{B-1}.$$

The last inequality comes from $\ln\,(n-1)! \le n \ln n - n + 1$ (see [7, p. 79]). But this reduces to

$$e^{B-1} < 2B^2 \ln B,$$

which is false for $B \ge 6$. For the remaining bases 3, 4 and 5, we can verify $x < B^B$ individually using (5).

Finally, upon multiplying (6) by $\ln x$ and using our result $\ln x < B \ln B$, we have

$$x \;<\; 2\,(B-1)!\,B^2 \ln B,$$

which is the desired bound.

We used UBASIC and a slightly sharpened form of the bound above to lists all of the solutions for various small bases, the result of this search is in Table 1.

Insert Table 1 near here

Alternately we could choose an integer x and ask if there is any base B for which the equation (4) has a solution. Clearly $x \geq B$. If we find the least integer n such that $n! \geq \pi(x)$, then we know $b_0 = (x \bmod B) \leq n$, so B is a divisor of $x-i$ for some $i \leq n$. For each x we then have a relative short list of possible bases. In this way we find all of the prime integers $x \leq 160{,}000{,}000$ such that (4) holds (x and B are written in base 10):

(x,B) = (3,2), (3,3), (5,2), (5,3), (17,14), (19,4), (19,8), (97,24), (97,93), (101,5), (103,9), (229,5), (661,132), (661,656), (673,334), (701,232), (5449,908), (5449,5443), (5501,7), (6473,1078), (6521,10), (6719,7), (6733,7), (49037,49030), (49043,24518), (49277,7039), (56809,9467), (64921,8), (114599,8), (484061,484053), (485909,60738), (495491,9), (560437,9), (5222447,5222438), (5222501,2611246), (5222837,1305707), (5224451,580494), (5224903,10), (5378437,15), (6480811,15), (61194733,61194723), (61285057,6128505), (62009933,11) and (67717891,7524209).

There are infinitely many such solutions! To see this, let p_n be the nth prime, then $(x,B) = (p_{n!+1}, p_{n!+1}-n)$ is a solution to (4).

The multifactorials

Instead of the factorial function, we could use the double factorial function $n!!$ [8, p. 258] or its generalization—the multifactorial function. These are defined for integers n as follows.

$n! = 1$	for $n \leq 1$,	otherwise	$n! = n \cdot (n-1)!$	(n factorial)
$n!! = 1$	for $n \leq 1$,	otherwise	$n!! = n \cdot (n-2)!!$	(n double-factorial)
$n!!! = 1$	for $n \leq 1$,	otherwise	$n!!! = n \cdot (n-3)!!!$	(n triple-factorial)

and in general

$n!_k = 1$	for $n \leq 1$,	otherwise	$n!_k = n \cdot (n-k)!_k$	(n k-factorial).

For example, $13!!! = 13!_3 = 13\cdot10\cdot7\cdot4\cdot1$ and $23!_4 = 23\cdot19\cdot15\cdot11\cdot7\cdot3$.

The approach above can also be used to bound the integers to check for the multi-factorials. Using the double factorial function, we have four solutions: 34, 6288, 10982, and 11978. For the triple factorial function, we have these four solutions: 45, 117, 127, and 2199. If we restrict ourselves to prime solutions, then there are only two additional solutions provided by all of the multifactorial functions:

$$\pi(127) = 1!!! + 2!!! + 7!!!$$

and

$$\pi(97) = 9!_7 + 7!_7.$$

Other functions

If we just count the digits, there is one solution: 2 ($\pi(2) = 1$, and 2 has 1 digit). If we add the digits then there are four solutions: 0, 15, 27, and 39 (none of which is prime). Using higher powers, we find the following prime solutions:

$$\pi(93701) = 9^4 + 3^4 + 7^4 + 0^4 + 1^4$$

$$\pi(1776839) = 1^5 + 7^5 + 7^5 + 6^5 + 8^5 + 3^5 + 9^5$$

$$\pi(1264061) = 1^6 + 2^6 + 6^6 + 4^6 + 0^6 + 6^6 + 1^6$$

$$\pi(\mathbf{34543}) = 3^3 + 4^4 + 5^5 + 4^4 + 3^3.$$

Note that 34543, found by the first author, is also palindromic [9].

Questions for the reader

Why add the terms corresponding to each digit? We could multiply:

$$\pi(1321) = 1^3 \cdot 3^3 \cdot 2^3 \cdot 1^3$$

or alternate signs:

$$\pi(19) = -1 + 9$$

$$\pi(53) = 5^2 - 3^2, \quad \pi(227) = 2^2 - 2^2 + 7^2, \quad \pi(929) = 9^2 - 2^2 + 9^2$$

$$\pi(47501) = -4! + 7! - 5! + 0! - 1!.$$

How about backwards exponentiation: $\pi(17) = 7^1$ and $\pi(23) = 3^2$?

Exploring other functions such as the sum of divisors function, may also prove interesting. In all such cases, the authors would be pleased to hear of your results.

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Table 1: Solutions in other bases

base B	solutions written in base 10 (primes in boldface)
2	**3**, **5**, 6, 8, 9, 10
3	**3**, 4, **5**, 6, 8
4	4, 6, 10, **19**, 27, 63
5	**101**, **229**, 374
6	18, 20, 134, 731, 737, 789, 1547
7	**5501**, 5690, 6530, **6719**, 6726, **6733**, 13180, 14395
8	**19**, 844, 5530, 13174, 49336, 49337, 58341, 58348, **64921**, 106108, **114599**
9	21, **103,** 364, 851, 105712, 105721, 105730, 493832, 494055, 494056, **495491,** 495524, 550620, 550622, 550654, **560437,** 1029375, 1029376, 1029459, 1031285, 1041084, 1041085, 1041128, 1041411
11	5704, 5715, 6705, 106022, 107114, 5456695, 5927793, 5927804, 5927815, 5927825, 16981728, 61924436, 61934787, **62009933**, 63370216, 67733027, 67733038, 129294118, 134549464, 134549475, 134549486, 134551268, 136058582, 136058583, 197958265